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Cylindrically symmetric distributions of matter and magnetic energy in equilibrium in general relativity

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Abstract. Solutions of the Einstein–Maxwell equations corresponding to cylindrically symmetric distributions of stressless conducting matter with an axial magnetic field have been found, which can be matched with an outside pure magnetic field solution originally due to Bonnor. It is also shown that Melvin's magnetic universe cannot be fitted with dust distributions in this way.

1. Introduction

Stationary solutions of Einstein's field equations (without the cosmological term) are already known in which the effect of gravitation is balanced by that of the rotation of matter (Einstein 1939, Raychaudhuri and Som 1962, Maitra 1966). Recently Som and Raychaudhuri (1968) have found a cylindrically symmetric solution for charged dust with rotation, where, however, the Lorentz force vanishes so that equilibrium is again due to the balancing of the gravitational effect of matter and electromagnetic field energy by the centrifugal action of rotation. Again Ozsvath (1967) has investigated homogeneous distributions of perfectly conducting matter and, although in his solutions the Lorentz force does not always vanish, still it turns out that there is always a non-vanishing vorticity. The question that we propose to investigate here is whether an equilibrium can be obtained even in the absence of rotation by the effect of electromagnetic interaction in uncharged matter.

In the following section we shall consider two classes of such solutions:

(i) Where the matter density ρ bears a linear relation to the magnetic field energy, viz. $8\pi\rho = bH^2$, where b is a positive constant. This class of solutions reduces to the known vacuum solution of Melvin (1964) when $b = 0$.

(ii) A second class of solutions which reduces in the limit $\rho \to 0$ to the more generalized solution of Bonnor (1954), Raychaudhuri (1960), Ghosh and Sengupta (1965).

In § 3 we shall find a solution which can be matched smoothly to the vacuum solution of Bonnor (1954). It will be further shown that Melvin's vacuum solution cannot be matched to that of a dust in this way.

2. Static solutions inside conducting dust

We consider the general static cylindrically symmetric line element

$$ds^2 = g_{11}\,dr^2 + g_{22}\,dz^2 + g_{33}\,d\phi^2 + g_{44}\,dt^2 \tag{1}$$

where r and z are the radial and axial coordinates, respectively, and ϕ is the angular coordinate. We number r, z, ϕ, t as 1, 2, 3 and 4, respectively. The $g_{\mu\nu}$ are functions of r alone.

We assume that the matter is at rest in the coordinate system of (1) so that the velocity $v^\mu = \delta^\mu_4/\sqrt{g_{44}}$. We further assume the existence of an axial magnetic field along the z direction and take the matter to be uncharged and perfectly conducting, so that the electric field vanishes. Hence, the only non-vanishing component of the electromagnetic field tensor is F^{31}. It is evident that, in this case, $R^3_3 + R^4_4 = 0$. We can therefore reduce the line element to the Weyl canonical form (Synge 1960, pp. 310–2)

$$ds^2 = e^{2\alpha}\,dt^2 - e^{2\beta-2\alpha}(dr^2 + dz^2) - r^2\,e^{-2\alpha}\,d\phi^2. \tag{2}$$

The Einstein–Maxwell equations are

$$R^{\mu}_{\nu}-\tfrac{1}{2}R\delta^{\mu}_{\nu} = -8\pi T^{\mu} \tag{3}$$

$$T^{\mu}_{\nu} = \rho v^{\mu}v_{\nu}+\frac{1}{4\pi}(\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^{\mu}_{\nu}-F^{\mu\alpha}F_{\nu\alpha}) \tag{4}$$

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^{\mu} \tag{5}$$

$$F_{[\mu\nu,\alpha]} = 0. \tag{6}$$

The magnetic field vector is defined by

$$H^{\alpha} = \frac{\frac{1}{2}\epsilon^{\alpha\mu\nu\beta}}{\sqrt{-g}}F_{\mu\nu}v_{\beta} \tag{7}$$

where $\epsilon^{\alpha\mu\nu\beta}$ is the Levi-Civita tensor density.

For the line element (2) we have the following three independent equations:

$$e^{2\alpha-2\beta}\left(\frac{\beta_1}{\gamma}-\alpha_1{}^2\right) = H^2 \tag{8}$$

$$e^{2\alpha-2\beta}(\beta_{11}+\alpha_1{}^2) = H^2 \tag{9}$$

$$e^{2\alpha-2\beta}\left(\alpha_1{}^2+\beta_{11}-2\alpha_{11}-\frac{2\alpha_1}{r}\right) = -8\pi\rho-H^2 \tag{10}$$

where $H^2 = -H^{\alpha}H_{\alpha}$ and subscript 1 indicates differentiation with respect to r.

The only non-vanishing component of the conduction current is in the ϕ direction and is given by

$$J^3 = \frac{4\pi}{\sqrt{-g}}(F^{31}\sqrt{-g})_{,1}. \tag{11}$$

We note, however, that the equation system (8)–(10) is undetermined as there are only three equations connecting four unknown variables (α, β, ρ, H^2). We can, therefore, adjoin a further relation between them. We shall first consider the case

$$8\pi\rho = bH^2 \tag{12}$$

where b is a constant not less than 0.

From equations (8) and (9) we obtain

$$\beta_{11}-\frac{\beta_1}{r} = -2\alpha_1{}^2. \tag{13}$$

From equations (9) and (10) we have

$$8\pi\rho = e^{2\alpha-2\beta}\left(2\alpha_{11}-2\beta_{11}+\frac{2\alpha_1}{r}-2\alpha_1{}^2\right). \tag{14}$$

This can be reduced to the form

$$(\alpha-b'\beta)_{,11}+\frac{1}{r}(\alpha-b'\beta)_{,1} = 0 \tag{15}$$

where

$$b' = \frac{b+2}{4}. \tag{16}$$

Equation (15) on integration yields either

$$\alpha-b'\beta = l \tag{17}$$

or

$$\alpha - b'\beta = c \ln \frac{r}{r_0} \tag{18}$$

where c, l and r_0 are constants of integration.

From (17) we obtain

$$\alpha = l + \frac{1}{2b'} \ln c(b'^2 r^2 + a) \tag{19}$$

$$\beta = \frac{1}{2b'^2} \ln c\,(b'^2 r^2 + a). \tag{20}$$

l can be reduced to zero by a suitable choice of coordinates. Further, we must take $ca = 1$ as otherwise there would be a singularity on the axis. This is because the circumference of the circle of radius r tends to $2\pi r(ca)^{-1/2b'}$ and the proper radius tends to $r(ca)^{(1-b')/2b'^2}$ as $r \to 0$. Hence, the solution is

$$\alpha = \frac{1}{2b'} \ln \left(1 + \frac{b'^2 r^2}{a}\right) \tag{21}$$

$$\beta = \frac{1}{2b'^2} \ln \left(1 + \frac{b'^2 r^2}{a}\right). \tag{22}$$

This solution is singularity free and reduces to Melvin's (1964) solution for $b = 0$ (i.e. $b' = \frac{1}{2}$)

$$\alpha = \ln \left(1 + \frac{k^2}{4} r^2\right) \tag{23}$$

$$\beta = 2 \ln \left(1 + \frac{k^2}{4} r^2\right) \tag{24}$$

where $k = 1/\sqrt{a} = F^{31}\sqrt{g}$.

The density, magnetic field and conduction current density are given by

$$8\pi\rho = \frac{b\, e^{2\alpha - 2\beta}}{a(1 + b'^2 r^2/a)^2} = \frac{b}{a} \left(1 + \frac{b'^2 r^2}{a}\right)^{(b' - 2b'^2 - 1)/b'^2}. \tag{25}$$

This is positive only when $a > 0$.

$$H^2 = \frac{8\pi\rho}{b} = \frac{e^{2\alpha - 2\beta}}{a(1 + b'^2 r^2/a)^2} = \frac{(1 + b'^2 r^2/a)^{(b' - 2b'^2 - 1)/b'^2}}{a} \tag{26}$$

$$J^3 = -\frac{2\pi b b'}{a^{3/2}} \frac{e^{3\alpha - 2\beta}}{(1 + b'^2 r^2/a)^2} = -\frac{2\pi b b'}{a^{3/2}} \left(1 + \frac{b'^2 r^2}{a}\right)^{(3b' - 4b'^2 - 2)/2b'^2}. \tag{27}$$

The integration of equation (18) leads to three different cases which we consider separately.

2.1. *Case* 1. $1 - 4cb' > 0$

$$\alpha = \frac{1}{2b'} \ln \left\{ r^{2d} + \frac{a(d+1)}{d-1} \right\} + \frac{1-d}{2b'} \ln r \tag{28}$$

where

$$d = \pm (1 - 4cb')^{1/2}$$

$$\beta = \frac{1}{b'} \alpha - \frac{c}{b'} \ln r + \ln B \tag{29}$$

where B is a constant of integration.

Further,

$$8\pi\rho = \frac{abd^2(d+1)}{b'^2(d-1)} e^{2\alpha-2\beta} r^{2d-2} \left\{ r^{2d} + \frac{a(d+1)}{d-1} \right\}^{-2} \tag{30}$$

$$J^3 = -\frac{\pi bd(d+1)}{b'^2} \left\{ \frac{a(d+1)}{d-1} \right\}^{1/2} r^{d-3}(r^{2d}-a)\, e^{3\alpha-2\beta} \left\{ r^{2d} + \frac{a(d+1)}{d-1} \right\}^{-2}. \tag{31}$$

For $b = 0$ the solutions reduce to

$$\alpha = \ln\left\{ r^{\lambda} + \frac{a(d+1)}{d-1} r^{2-\lambda} \right\} = \ln\left\{ r^{\lambda} + \frac{k^2}{4(\lambda-1)^2} r^{2-\lambda} \right\} \tag{32}$$

where $\lambda = 1+d$

$$\beta = 2\alpha + \lambda(\lambda-2)\ln r + \ln B. \tag{33}$$

The pure magnetic field solution (32), (33) was given in this form by Ghosh and Sengupta (1965). The solutions obtained earlier by Bonnor (1954) and Raychaudhuri (1960) can be reduced to this form by suitable transformations. The solutions have a singularity on the z axis. When $\lambda = 0$ the solutions (32) and (33) reduce to the form (23) and (24).

2.2. *Case* 2. $1-4cb' < 0$

$$\alpha = \frac{1}{2b'} \ln r(d\sin\theta + \cos\theta) \tag{34}$$

where

$$d = (4cb'-1)^{1/2}, \qquad \theta = \frac{d}{2}\ln\left(\frac{b'^2r^2}{a}\right)$$

$$\beta = \frac{1}{b'}\alpha - \frac{c}{b'}\ln r + \ln B \tag{35}$$

$$8\pi\rho = -\frac{e^{2\alpha-2\beta} bd^2(d^2+1)}{4b'^2r^2(d\sin\theta+\cos\theta)^2}. \tag{36}$$

In this case ρ and H^2 are both negative and the solution has no physical significance.

2.3. *Case* 3. $1-4cb' = 0$

$$\alpha = \frac{1}{2b'}[\ln r + \ln\{\ln(ar^2)-2\}] \tag{37}$$

$$\beta = \frac{1}{4b'^2}[\ln r + 2\ln\{\ln(ar^2)-2\}] + \ln B \tag{38}$$

$$8\pi\rho = -\frac{b\, e^{2\alpha-2\beta}}{b'^2r^2\{\ln(ar^2)-2\}^2}. \tag{39}$$

Again ρ and H^2 are both negative.

3. Fitting the solutions to an outside pure magnetic field solution

The solutions given by Bonnor (1954) and Melvin (1964) have a sourceless magnetic field along the z axis. We shall try to introduce a source in the form of a conduction current in the ϕ direction inside a perfectly conducting dust. Som (1968) has taken a rotating (but without net angular momentum) dust distribution with an axial magnetic field and matched the solution to that of Bonnor. But the magnetic field is source free in his case too.

First we note that the solution given by equations (21) and (22) can be matched with the exterior solutions (32) and (33) at $r = r_0$ if the following boundary conditions are satisfied:

$$\left(1+\frac{b'^2 r_0^2}{a}\right)^{1/2b'} = r_0^{\lambda}+\frac{k^2}{4(\lambda-1)^2} r_0^{2-\lambda} \tag{40}$$

$$\left(1+\frac{b'^2 r_0^2}{a}\right)^{1/2b'^2} = B r_0^{\lambda(\lambda-2)}\left\{r_0^{\lambda}+\frac{k^2}{4(\lambda-1)^2} r_0^{2-\lambda}\right\}^2 \tag{41}$$

$$\frac{b' r_0}{a+b'^2 r_0^2} = \frac{\lambda r_0^{\lambda-1}+\{k^2/4(\lambda-1)^2\}(2-\lambda) r_0^{1-\lambda}}{r_0^{\lambda}+\{k^2/4(\lambda-1)^2\} r_0^{2-\lambda}}. \tag{42}$$

$$\frac{r_0}{a+b'^2 r_0^2} = \frac{2\lambda r_0^{\lambda-1}+\{k^2/2(\lambda-1)^2\}(2-\lambda) r_0^{1-\lambda}}{r_0^{\lambda}+\{k^2/4(\lambda-1)^2\} r_0^{2-\lambda}}+\frac{\lambda(\lambda-2)}{r_0}. \tag{43}$$

These four equations may be used to determine the four unknown constants λ, k, a and B.

Secondly we shall consider another solution which can be nicely matched with the exterior solution. Since the equations (8)–(10) are underdetermined we may write for the interior metric

$$\alpha = \ln\left(k_1+k_2\frac{r^2}{a^2}\right) \tag{44}$$

where k_1 and k_2 are constants and a is the radius of the cylinder containing perfectly conducting dust. From the field equations we have

$$\beta = 2\ln\left(k_1+k_2\frac{r^2}{a^2}\right)+\frac{br^2}{2}+c. \tag{45}$$

The exterior metric is given by (32) and (33). Hence we have, from the boundary conditions,

$$k_1 = \frac{1}{2}\left\{(2-\lambda)a^{\lambda}+\frac{k^2}{4(\lambda-1)^2}\lambda a^{2-\lambda}\right\} \tag{46}$$

$$k_2 = \frac{1}{2}\left\{\lambda a^{\lambda}+\frac{k^2}{4(\lambda-1)^2}(2-\lambda)a^{2-\lambda}\right\} \tag{47}$$

$$b = \frac{1}{a^2}\lambda(\lambda-2) \tag{48}$$

$$c = \tfrac{1}{2}\lambda(\lambda-2)(2\ln a-1)+\ln B. \tag{49}$$

Now we have from equation (14)

$$8\pi\rho = -\frac{2b\,\mathrm{e}^{-(br^2+2c)}}{(k_1+k_2 r^2/a^2)^2}. \tag{50}$$

For ρ to be positive $b < 0$ and hence from (48) we obtain $0 < \lambda < 2$.

Further,

$$H^2 = \frac{\exp\{-(br^2+2c)\}}{(k_1+k_2 r^2/a^2)^2}\left\{\frac{4k_1k_2/a^2}{(k_1+k_2 r^2/a^2)^2}+b\right\}. \tag{51}$$

This is always positive if

$$\frac{4k_1k_2/a^2}{(k_1+k_2)^2} > -b.$$

If we substitute the values from equations (46)–(48), this condition reduces to

$$(\lambda-1)^2 > 0.$$

We exclude the case $\lambda = 1$ because the exterior metric becomes singular for this value.

Further, if the metric is not singular on the z axis then

$$c = -2 \ln k_1. \tag{52}$$

From (49) and (51) we have

$$\ln B = \tfrac{1}{2}\lambda(\lambda-2)(1-2\ln a)-2\ln k_1. \tag{53}$$

Thus all the constants have been determined.

If, however, $\lambda = 0$ or 2, $b = 0$ and hence from (50) $\rho = 0$. The metric everywhere is that given by Melvin.

We shall now show that the Melvin universe cannot be matched with matter dust in this way. Let us consider a cylinder of matter of radius a which is matched with the pure magnetic field solution on the exterior. For a static metric R^4_4 can be expressed in the following form:

$$2R^4_4\sqrt{-g} = \left(\frac{g^{\alpha\beta}\sqrt{-g}\,g_{44,\beta}}{g_{44}}\right)_{,\alpha}. \tag{54}$$

Further, from the form of our metric we have

$$2R^2_2\sqrt{-g} = \left(\frac{g^{\alpha\beta}\sqrt{-g}\,g_{22,\beta}}{g_{22}}\right)_{,\alpha}. \tag{55}$$

From the field equations we have

$$2(R^2_2-R^4_4)\sqrt{-g} = 16\pi\rho\sqrt{-g}. \tag{56}$$

Integrating this over a unit length of the cylinder of radius a, we obtain

$$\int_0^a\int_z^{z+1}\int_0^{2\pi} 16\pi\rho\sqrt{-g}\,dr\,dz\,d\phi = \int_z^{z+1}\int_0^{2\pi}\left(\frac{g^{11}\sqrt{-g}\,g_{22,1}}{g_{22}} - \frac{g^{11}\sqrt{-g}\,g_{44,1}}{g_{44}}\right)\bigg|_{r=a} dz\,d\phi.$$

Substituting the values from (32) and (33) we obtain the mass inside the cylinder as

$$m = \int_0^a\int_z^{z+1}\int_0^{2\pi}\rho\sqrt{-g}\,dr\,dz\,d\phi = \tfrac{1}{4}\lambda(2-\lambda). \tag{57}$$

m vanishes when $\lambda = 0$ or 2, so that a non-vanishing ρ cannot everywhere be positive. This proves our proposition.

4. Concluding remarks

We have exhibited above a class of solutions where the electromagnetic interaction is balanced by the gravitational field. Since the vorticity, shear and expansion are absent we may write from the definition of the Ricci tensor

$$R_{\mu\sigma}v^\mu v^\sigma = -\pi^\mu{}_{;\mu} \tag{58}$$

where

$$\pi^\mu = v^\mu{}_{;\nu}v^\nu. \tag{59}$$

Further, the divergence relation $T^{\mu\nu}{}_{;\nu} = 0$ gives from equation (4)

$$\pi^\mu = -\frac{1}{\rho}F^\mu{}_\nu J^\nu. \tag{60}$$

Hence we have, from the field equations,

$$\frac{J^2}{\rho}+\frac{1}{2\rho}F^{\mu\nu}(J_{\nu,\mu}-J_{\mu,\nu})+\left(\frac{1}{\rho}\right)_{,\mu}F^\mu{}_\nu J^\nu = -4\pi\rho - H^2 \tag{61}$$

where $J^2 = J_\nu J^\nu$.

The left-hand side gives the electromagnetic interaction and the effect of gravitation is given by the right-hand side.

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References

BONNOR, W. B., 1954, *Proc. Phys. Soc.* A, **67,** 225.
EINSTEIN, A., 1939, *Ann. Math., Lpz.*, **40,** 922–36.
GHOSH, R., and SENGUPTA, R., 1965, *Nuovo Cim.*, **38,** 1579.
MAITRA, S. C., 1966, *J. Math. Phys.*, **7,** 1025–30.
MELVIN, M. A., 1964, *Phys. Lett.*, **8,** 65.
RAYCHAUDHURI, A. K., 1960, *Ann. Phys.*, **11,** 501–9.
RAYCHAUDHURI, A. K., and SOM, M. M., 1962, *Proc. Camb. Phil. Soc.,* **58,** 338–45.
OZSVATH, I., 1967, *J. Math. Phys.*, **8,** 326–44.
SOM, M. M., 1968, *J. Phys.* A (*Proc. Phys. Soc.*), [2], **1,** 108–11.
SOM, M. M., and RAYCHAUDHURI, A. K., 1968, *Proc. R. Soc.* A, **304,** 81–6.
SYNGE, J. L., 1960, *Relativity, The General Theory* (Amsterdam: North-Holland).